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The method of multiple scales and non-linear dispersive waves

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The method of multiple scales is used to analyze three non-linear physical systems which support dispersive waves. These systems are (i) waves on the interface between a liquid layer and a subsonic gas flowing parallel to the undisturbed interface, (ii) waves on the surface of a circular jet of liquid, and (iii) waves in a hot electron plasma. It is found that the partial differential equations that govern the temporal and spatial variations of the wave-numbers, amplitudes, and phases have the same form for all of these systems. The results show that the non-linear motion affects only the phase. For the constant wave-number case, the general solution for the amplitude and the phase can be obtained.

1. Introduction

We consider in this paper three weakly non-linear physical systems which support dispersive waves whose amplitudes, phases, frequencies, and wave-numbers are slowly-varying functions of both space and time. These systems are (i) interaction of capillary gravity waves with a subsonic flow moving uniformly parallel to the undisturbed liquid surface, (ii) waves on the surface of a circular column of liquid neglecting gravity, and (iii) waves in a hot electron plasma.

A number of techniques have been used to treat such weakly non-linear dispersive waves. We limit our discussion below to those which treat both the temporal and spatial variations of the wave parameters. Sturrock (1957) used the derivative-expansion method (a form of the method of multiple scales) to determine the amplitude and phase variations of waves in non-linear electron plasmas. The method of multiple scales has been used by the following: Nayfeh (1965) to determine the amplitude and phase variation of waves in a hot electron plasma; Luke (1966) to determine the amplitude, frequency, and wave-number variations for waves governed by the Klein–Gordon equation as well as by a general variational equation of second order; McGoldrick (1970) to determine the amplitude and phase variations for the problem of second-harmonic resonance in the interaction of capillary and gravity waves; Emery (1970) to extend the work of Luke to the cases of several dependent variables, and several rapidly rotating phases; Nayfeh (1971 *a*) to determine the amplitude and phase variations for the

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problem of third-harmonic resonance in the interaction of capillary and gravity waves; and Nayfeh (1971*b*) to extend the work of McGoldrick by including the effects of near resonance, depth of the liquid layer, and an external subsonic gas flowing parallel to the undisturbed liquid surface.

Montgomery & Tidman (1964) and Tidman & Stainer (1965) used an extension of the Krylov–Bogoliubov–Mitropolski technique (Bogoliubov & Mitropolski 1961) to determine the amplitude and phase variations for waves in non-linear cold and hot electron plasmas, respectively. Whitham (1965*a*) used the method of averaging to determine the amplitude, frequency, and wave-number variations for waves governed by the Klein–Gordon equation, shock waves, and gravity waves.

Whitham (1965*b*) showed that his previous results (Whitham 1965*a*) could be obtained in a simpler and more elegant manner by averaging of the Lagrangian of the original system of equations, because the desired equations for the wave parameters are the Euler–Lagrange equations for the averaged Lagrangian. A similar technique has been in use for a long time by celestial mechanicians (see, for example, Brouwer & Clemence 1961) to determine the amplitude and phase variations for non-linear systems. According to this technique the Hamiltonian is averaged to remove the short period terms (rapidly rotating phase variation), and then the desired equations for the amplitude and phase variations are just Hamilton's equations corresponding to the averaged Hamiltonian. Whitham (1967*a*) applied the averaging of the Lagrangian technique to gravity waves, and Whitham (1967*b*) reviewed variational methods and their applications to water waves. Simmons (1969) used this technique to determine the variations of the amplitudes and the phases for the problem of triad resonance in the interaction of capillary and gravity waves. Grimshaw (1970) used this averaging technique to analyze solitary waves in water of variable depth.

The interaction of capillary-gravity waves with an external subsonic gas is treated in the next section. Capillary waves on a cylindrical column of liquid are then treated in § 3 while the problem of non-linear hot electron plasma oscillations is treated in § 4.

2. Interaction of capillary-gravity waves with a subsonic gas

In this section we consider non-linear waves on the interface between an inviscid liquid and an inviscid subsonic gas flowing with a uniform velocity U_g parallel to this interface. The gas is assumed to be of infinite depth, while the liquid is assumed to be of finite depth with its second face adjacent to a solid surface. The motion is assumed to be two-dimensional and to be represented by potential functions. Distances and time are made dimensionless using

$$k_c^{-1} = (\sigma/\rho g)^{\frac{1}{2}} \quad \text{and} \quad (gk_c)^{-\frac{1}{2}},$$

where g is the body acceleration assumed to be acting toward the liquid, and ρ and σ are the liquid's density and surface tension respectively. The gas density ρ_g is assumed to be small compared to the liquid density so that the gas body force can be neglected. Moreover, the gas velocity U_g is assumed to be very much larger than the surface wave velocity, so that the transient motion of the gas can be neglected.

A Cartesian co-ordinate system x, y is introduced such that the x axis is in the plane of the undisturbed interface while the y axis is normal to this interface and directed from the liquid to the gas. The potential functions representing the motions of the liquid and the gas are taken to be

$$g^{\frac{1}{2}}k_c^{-\frac{3}{2}}\phi(x,y,t), \quad U_g[x+\Phi(x,y,t)]/k_c$$

where the dimensionless functions ϕ and Φ are given by

$$\nabla^2\phi = 0, \quad -h < y \leqslant \eta \tag{2.1}$$

and (see, for example, Van Dyke 1964, p. 107)

$$\begin{aligned}\Phi_{yy}+m^2\Phi_{xx} = M^2[\tfrac{1}{2}(\gamma-1)(2\Phi_x+\Phi_x^2+\Phi_y^2)(\Phi_{xx}+\Phi_{yy})\\ +(2\Phi_x+\Phi_x^2)\Phi_{xx}+2(1+\Phi_x)\Phi_y\Phi_{xy}+\Phi_y^2\Phi_{yy}] \quad (\eta \leqslant y < \infty)\end{aligned} \tag{2.2}$$

for $-\infty < x < \infty$, where $\eta(x,t)$ is the elevation of the wave above the undisturbed interface, M is the gas Mach number, and

$$m^2 = 1-M^2.$$

Away from the gas/liquid interface,

$$\phi_y = 0 \quad \text{at} \quad y = -h, \tag{2.3}$$

$$\Phi_y \to 0 \quad \text{as} \quad y \to \infty, \tag{2.4}$$

where h is the dimensionless depth of the undisturbed liquid layer. At the gas/liquid interface, the normal components of the gas and the liquid velocities are equal to each other and to that of the interface itself; that is

$$\eta_t+\phi_y = \eta_x\phi_x \quad \text{at} \quad y = \eta, \tag{2.5}$$

$$\eta_x-\Phi_y = -\eta_x\Phi_x \quad \text{at} \quad y = \eta. \tag{2.6}$$

The remaining boundary condition is provided by the balance of the normal forces at the interface; that is,

$$\eta-\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2) = \eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}}-\tfrac{1}{2}m\chi C_p \quad \text{at} \quad y = \eta \tag{2.7}$$

where $\chi = \rho_g U_g^2/(\rho g\sigma)^{\frac{1}{2}}$ and C_p is the pressure perturbation coefficient exerted by the gas on the interface due to the appearance of waves, and it is given by (see, for example, Liepmann & Roshko 1960, p. 206)

$$C_p = (2/\gamma M^2)\{[1-\tfrac{1}{2}(\gamma-1)M^2(2\Phi_x+\Phi_x^2+\Phi_y^2)]^{\gamma/\gamma-1}-1\}, \tag{2.8}$$

with γ the specific heat ratio of the gas.

To determine an approximate solution for (2.1)–(2.8) for small but finite amplitudes using the method of multiple scales, we assume that

$$\eta(x,t) = \sum_{n=1}^{3}\epsilon^n\eta_n(\xi,\tau,\theta)+O(\epsilon^4), \tag{2.9}$$

$$\phi(x,y,t) = \sum_{n=1}^{3}\epsilon^n\phi_n(\xi,\tau,\theta,y)+\epsilon^2\bar{\phi}_2(t,\xi,\tau)+O(\epsilon^4), \tag{2.10}$$

$$\Phi(x,y,t) = \sum_{n=1}^{3}\epsilon^n\Phi_n(\xi,\tau,\theta,y)+O(\epsilon^4), \tag{2.11}$$

where ϵ is a small parameter of the order of the maximum steepness ratio of the wave, and

$$\xi = \epsilon^2 x, \quad \tau = \epsilon^2 t, \quad \theta = \zeta(\xi,\tau)/\epsilon^2. \tag{2.12}$$

Thus, ξ and τ are slow scales, whereas θ is a fast scale. Since (2.3) and (2.4) are linear, each ϕ_n satisfies (2.3) while each Φ_n satisfies (2.4). The function $\bar{\phi}_2$ corresponds to absorbing the Bernoulli constant into the potential of the liquid. The derivatives are transformed according to

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta} + \epsilon^2 \frac{\partial}{\partial \xi}, \tag{2.13a}$$

$$\frac{\partial^2}{\partial x^2} = k^2 \frac{\partial^2}{\partial \theta^2} + 2\epsilon^2 k \frac{\partial^2}{\partial \theta\, \partial \xi} + \epsilon^2 k_\xi \frac{\partial}{\partial \theta} + \epsilon^4 \frac{\partial^2}{\partial \xi^2}, \tag{2.13b}$$

$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \theta} + \epsilon^2 \frac{\partial}{\partial \tau}, \tag{2.13c}$$

where
$$k = \zeta_\xi \quad \text{and} \quad \omega = -\zeta_\tau. \tag{2.13d}$$

Substituting (2.9)–(2.13) into (2.1)–(2.8), and equating coefficients of like powers of ϵ, we get equations to determine η_n, ϕ_n, and Φ_n. The solution of the first-order problem is taken to be

$$\eta_1 = A(\xi,\tau)e^{i\theta} + \bar{A}(\xi,\tau)e^{-i\theta}, \tag{2.14}$$

$$\phi_1 = \frac{i\omega}{k}[A(\xi,\tau)e^{i\theta} - \bar{A}(\xi,\tau)e^{-i\theta}]\frac{\cosh k(y+h)}{\sinh kh}, \tag{2.15}$$

$$\Phi_1 = -\frac{i}{m}[A(\xi,\tau)e^{i\theta} - \bar{A}(\xi,\tau)e^{-i\theta}]e^{-kmy}, \tag{2.16}$$

where A is a complex unknown function of ξ and τ, and ω and k satisfy the dispersion relationship

$$\omega^2 = k(k^2 - k\chi + 1)/C_1, \quad C_n = \coth nkh. \tag{2.17}$$

Then, the solution of the second-order problem is

$$\eta_2 = 2f_1 A^2 e^{2i\theta} + \text{CC}, \tag{2.18}$$

$$\phi_2 = 2i\frac{\omega}{k} f_2 A^2 e^{2i\theta}\frac{\cosh 2k(y+h)}{\sinh 2kh} + \text{CC}, \tag{2.19}$$

$$\Phi_2 = -\frac{i}{m}\left[2f_3 + \tfrac{1}{4}k(\gamma+1)\frac{M^4}{m^2}y\right] A^2 e^{2i\theta}e^{-2kmy} + \text{CC}, \tag{2.20}$$

where CC stands from complex conjugate, and

$$f_1 = \frac{\omega^2(3 - C_1^2 - 4C_1C_2) + 2k^2\chi[m^2 + \frac{1}{4}(\gamma+1)M^4]/m^3}{4} \times [4k^2 - 2k\chi + 1 - 2(k^2 - k\chi + 1)C_2/C_1], \tag{2.21}$$

$$f_2 = f_1 - \tfrac{1}{2}C_1 k, \tag{2.22}$$

$$f_3 = f_1 + \left[\frac{1}{4}\frac{1+m^2}{m^2} + \tfrac{1}{16}(\gamma+1)\frac{M^4}{m^3}\right]k. \tag{2.23}$$

With the first- and second-order solutions known, the third-order problem becomes

$$k^2\phi_{3\theta\theta}+\phi_{3yy} = \left\{\left[2k\left(\frac{\omega A}{k\sinh kh}\right)_\xi+\frac{\omega A}{k\sinh kh}k_\xi\right]\cosh k(y+h)\right.$$
$$\left.+\frac{\omega A}{\sinh kh}k_\xi(y+h)\sinh k(y+h)\right\}e^{i\theta}+\mathrm{CC}, \quad (2.24)$$

$$k^2m^2\Phi_{3\theta\theta}+\Phi_{3yy} = -4i(p_1+p_2y)A^2\bar{A}e^{i\theta}e^{-3kmy}$$
$$-m[2kA_\xi+k_\xi A-2mkk_\xi Ay]e^{i\theta}e^{-kmy}+\mathrm{CC}+\mathrm{NSPT}, \quad (2.25)$$

$$-\omega\eta_{3\theta}+\phi_{3y} = -(4ip_3A^2\bar{A}+A_\tau)e^{i\theta}+\mathrm{CC}+\mathrm{NSPT} \quad \text{at} \quad y=0, \quad (2.26)$$

$$k\eta_{3\theta}-\Phi_{3y} = -(4ip_4A^2\bar{A}+A_\xi)e^{i\theta}+\mathrm{CC}+\mathrm{NSPT} \quad \text{at} \quad y=0, \quad (2.27)$$

$$\eta_3+\omega\phi_{3\theta}-k^2\eta_{3\theta\theta}-mk\chi\Phi_{3\theta} = \left[4(p_5+\chi p_6)A^2\bar{A}+i\left(\frac{\omega A}{k\sinh kh}\right)_\tau\cosh kh\right.$$
$$\left.+i\frac{\omega}{k}hk_\tau A+i(2k-\chi)A_\xi+ik_\xi A\right]e^{i\theta}+\mathrm{CC}+\mathrm{NSPT} \quad \text{at} \quad y=0, \quad (2.28)$$

where NSPT stands for non-secular producing terms, and the p's are defined in the appendix.

The particular solution of (2.24)–(2.28) contains secular terms which make η_3/η_1 be unbounded as $\theta\to\infty$. The condition which must be satisfied for there to be no secular terms is

$$2A_\tau+2c_gA_\xi+c_g'Ak_\xi = 8iJA^2\bar{A}, \quad (2.29)$$

where $c_g = d\omega/dk$ is the group velocity, $c_g' = dc_g/dk$, and

$$J = \frac{k}{2C_1\omega}\left[-\frac{C_1\omega p_2}{k}-\frac{1}{4}\frac{\chi p_1}{km}-\frac{\chi p_2}{16k^2m^2}+\chi p_4+\chi p_6+p_5\right]. \quad (2.30)$$

If we let $A = \frac{1}{2}a\exp i\beta$ with real a and β, the solution to second order is

$$\eta = \epsilon a\cos(\theta+\beta)+\epsilon^2a^2f_1\cos 2(\theta+\beta)+O(\epsilon^3), \quad (2.31)$$

$$\phi = -\epsilon a\omega\sin(\theta+\beta)\frac{\cosh k(y+h)}{k\sinh kh}-\epsilon^2a^2\omega f_2\sin 2(\theta+\beta)\frac{\cosh 2k(y+h)}{k\sinh 2kh}+O(\epsilon^3), \quad (2.32)$$

$$\Phi = \frac{\epsilon a}{m}\sin(\theta+\beta)e^{-kmy}+\frac{\epsilon^2a^2}{m}\left[f_3+\tfrac{1}{8}k(\gamma+1)\frac{M^4}{m^2}y\right]\sin 2(\theta+\beta)e^{-2kmy}+O(\epsilon^3), \quad (2.33)$$

where

$$k = \theta_x, \quad \omega = -\theta_t, \quad (2.34)$$

$$(\partial k/\partial\tau)+c_g(\partial k/\partial\xi) = 0, \quad (2.35)$$

$$\omega^2 = k(k^2-k\chi+1)\tanh kh, \quad (2.36)$$

$$(\partial a^2/\partial\tau)+\partial(c_ga^2)/\partial\xi = 0, \quad (2.37)$$

$$(\partial\beta/\partial\tau)+c_g(\partial\beta/\partial\xi) = Ja^2. \quad (2.38)$$

If ω and k are constants, and if a and β are independent of ξ, than a = a constant and $\beta = Ja^2\tau$. In this case, the above solution reduces to that obtained by Nayfeh & Saric (1971). If, in addition, $\chi = 0$ (no external gas), the above solution reduces to those obtained by Kamesvara Rav (1920), Barakat & Houston (1968), and

Nayfeh (1970*a*). As $h \to \infty$ also (infinite depth), the solution reduces to those of Wilton (1915) and Pierson & Fife (1961).

For an infinite liquid depth (i.e. $h \to \infty$),

$$f_1 = \frac{1}{2}\frac{\omega^2 - k^2\chi[m^2 + \frac{1}{4}(\gamma+1)M^4]/m^3}{1-2k^2}, \tag{2.39}$$

$$f_2 = f_1 - \tfrac{1}{2}k, \tag{2.40}$$

$$f_3 = f_1 + \tfrac{1}{4}k\left[\frac{1+m^2}{m} + \tfrac{1}{4}(\gamma+1)\frac{M^4}{m^3}\right], \tag{2.41}$$

$$\omega^2 = k(k^2 - k\chi + 1). \tag{2.42}$$

Equations (2.31) and (2.33) remain unchanged in form while (2.32) becomes

$$\phi = -\epsilon a(\omega/k)\sin(\theta+\beta)\,e^{ky} - \epsilon^2 a^2 f_2(\omega/k)\sin 2(\theta+\beta)\,e^{2ky} + O(\epsilon^3). \tag{2.43}$$

In the absence of the external gas (i.e. $\chi = 0$),

$$f_1 = \frac{k\omega^2(3 - C_1^2 - 4C_1C_2)}{4[k(4k^2+1) - 2\omega^2 C_2]}, \quad f_2 = f_1 - \tfrac{1}{2}C_1 k, \tag{2.44}$$

$$\omega^2 = k(k^2+1)\tanh kh. \tag{2.45}$$

In this case, η and ϕ are still given by (2.31) and (2.32) while $\Phi = 0$. If, in addition, the liquid is infinite in depth,

$$\eta = \epsilon a\cos(\theta+\beta) + \tfrac{1}{2}k\frac{1+k^2}{1-2k^2}\epsilon^2 a^2\cos 2(\theta+\beta) + O(\epsilon^3), \tag{2.46}$$

$$\phi = -\epsilon a\frac{\omega}{k}\sin(\theta+\beta)\,e^{ky} - \tfrac{3}{2}\epsilon^2 a^2\omega\frac{k^2}{1-2k^2}\sin 2(\theta+\beta)\,e^{2ky} + O(\epsilon^3), \tag{2.47}$$

$$\omega^2 = k(k^2+1). \tag{2.48}$$

In addition, if ω and k are constants, the solution (2.46)–(2.48) with a and β given by (2.37) and (2.38) is the same as that obtained by Nayfeh (1971*a*).

Although the solution (2.31)–(2.38) is valid for a wide range of values of ω and k, it breaks down when

$$k(4k^2 - 2k\chi + 1)\tanh 2kh = 2(k^2 - k\chi + 1)\tanh kh. \tag{2.49}$$

This condition represents the second-harmonic resonant case treated by Nayfeh (1971*b*) for constant ω and k but varying a and β. The second-harmonic resonant case of infinite liquid depth and no external gas has been treated by Simmons (1969) and McGoldrick (1970). Periodic wave solutions for this resonant condition were obtained by Nayfeh & Saric (1971), Barakat & Houston (1968) for $\chi = 0$, and Wilton (1915) and Pierson & Fife (1961) for $\chi = 0$ and $h \to \infty$.

Equations (2.35), (2.37), and (2.38) have the same form as those which will be obtained in §§3 and 4 for waves on a circular column of liquid, and for waves in a hot electron plasma. Therefore, the discussion of these equations is presented for the three problems discussed in this paper in §5.

3. Waves on the surface of a cylindrical jet of fluid

It is assumed that the fluid jet is both inviscid and incompressible, and the effects of the surrounding fluid are negligible. The flow is assumed to start from rest so that it can be represented by a potential function $\tilde{\phi}$, and the undisturbed jet is assumed to be circular with radius R. The following analysis is restricted to axisymmetric waves.

All physical quantities are made dimensionless using the characteristic length R, and the characteristic time $(\rho R^3/T)^{\frac{1}{2}}$, where ρ and T are the fluid density and surface tension respectively. In a dimensionless cylindrical co-ordinate system (r and x with x along the jet axis), the dimensionless potential function ϕ is given by

$$\nabla^2\phi = 0, \tag{3.1}$$

for $\quad r \leqslant 1+\eta(x,t) \quad \text{and} \quad -\infty < x < \infty,$

where η is the dimensionless elevation in the r direction, and t is the dimensionless time. The kinematic and dynamic boundary conditions at the liquid surface are

$$\eta_t + \phi_r = \eta_x \phi_x, \tag{3.2}$$

$$\phi_t - \tfrac{1}{2}(\phi_x^2+\phi_r^2) + \eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} - (1+\eta)^{-1}(1+\eta_x^2)^{-\frac{1}{2}} + 1 = 0$$
$$\text{at} \quad r = 1+\eta(x,t). \tag{3.3}$$

To determine an approximate solution to (3.1)–(3.3) we assume that

$$\eta(x,t) = \sum_{n=1}^{3} \epsilon^n \eta_n(\xi,\tau,\theta) + O(\epsilon^4), \tag{3.4}$$

$$\phi(x,r,t) = \sum_{n=1}^{3} \epsilon^n \phi_n(\xi,\tau,\theta,r) + O(\epsilon^4), \tag{3.5}$$

where ϵ is a small but finite amplitude quantity which is of the order of the maximum steepness ratio of the surface waves, and

$$\xi = \epsilon^2 x, \quad \tau = \epsilon^2 t, \quad \theta = \zeta(\xi,\tau)/\epsilon^2. \tag{3.6}$$

The transformation of the x and t derivatives are the same as in (2.13). Substituting (3.4)–(3.6) into (3.1)–(3.3) and equating coefficients of like powers of ϵ, we obtain equations to determine η_n and ϕ_n.

The solution of the first-order problem is taken to be

$$\eta_1 = A(\xi,\tau)e^{i\theta} + \bar{A}(\xi,\tau)e^{-i\theta}, \tag{3.7}$$

$$\phi_1 = i(\omega/k)\,[A(\xi,\tau)e^{i\theta} - \bar{A}(\xi,\tau)e^{-i\theta}]\,[I_0(kr)/I_1(k)], \tag{3.8}$$

where ω and k satisfy the dispersion relationship

$$\omega^2 = k(k^2-1)\,[I_1(k)/I_0(k)]. \tag{3.9}$$

Here, I_0 and I_1 are the modified Bessel functions of the zero and first order, respectively. If ω, k, and A are assumed to be constants, (3.7)–(3.9) reduce to the solution of Rayleigh (1945, p. 351). Travelling-wave solutions are possible only when $k > 1$. If $k < 1$, disturbances grow with time, and hence the liquid column is unstable. In what follows, we restrict our analysis to the case $k > 1$. With the above first-order solution, the solution of the second-order problem is

$$\eta_2 = q_1 A\bar{A} + q_3 A^2 e^{2i\theta} + \text{CC}, \tag{3.10}$$

$$\phi_2 = iq_2 \omega A^2 [I_0(2kr)/2kI_1(2k)] e^{2i\theta} + \text{CC}, \tag{3.11}$$

where

$$q_1 = \tfrac{1}{2}\omega^2 [I_0(k)/I_1(k)] - \tfrac{1}{2}\omega^2 + 1 - \tfrac{1}{2}k^2, \tag{3.12}$$

$$q_2 = \frac{3k[(1-3k^2)[I_0(k)/I_1(k)] + (k^2-1)[I_1(k)/I_0(k)] + k - (1/k)]}{4k^2 - 1 - 2\omega^2 [I_0(2k)/kI_1(2k)]}, \tag{3.13}$$

$$q_3 = \tfrac{1}{2}q_2 + [kI_0(k)/I_1(k)] - \tfrac{1}{2}. \tag{3.14}$$

If the initial conditions are periodic in x, and if the motion is started from rest, then $q_1 = -\frac{1}{2}$.

The above first- and second-order solutions determine the third-order problem. Its particular solution contains secular terms which make η_3/η_1 unbounded as $\theta \to \infty$ unless

$$2\frac{\partial A}{\partial \tau} + 2c_g \frac{\partial A}{\partial \xi} + c_g' k_\xi A = 8iJA^2\bar{A}, \tag{3.15}$$

where $c_g = d\omega/dk$, $c_g' = dc_g/dk$, and

$$J = -\tfrac{1}{8}\omega \left[Q_1 + \frac{kI_1(k)}{I_0(k)} Q_2 \right]. \tag{3.16}$$

As in §3.1, letting $A = \frac{1}{2}a \exp i\beta$ with real a and β, and separating real and imaginary parts in (3.15), we obtain

$$\frac{\partial a^2}{\partial \tau} + \frac{\partial}{\partial \xi}(c_g a^2) = 0, \tag{3.17}$$

$$\frac{\partial \beta}{\partial \tau} + c_g \frac{\partial \beta}{\partial \xi} = Ja^2. \tag{3.18}$$

If we consider the temporal variation only, ω, k, and a are constants, and $\beta = Ja^2\tau$. This latter solution agrees with that of Wang (1968) if $q_1 = -\frac{1}{2}$. Since $J \to \infty$ as $k \to 1$, the solution presented in this section is invalid near $k = 1$. Nayfeh (1970*b*) obtained an expansion valid near $k = 1$, taking into account the temporal variation only. Equations (3.17) and (3.18) have the same form as (2.37) and (2.38) and they will be discussed in §5.

4. Non-linear oscillations in a hot electron plasma

We consider in this section non-linear longitudinal travelling waves in a hot electron plasma in the absence of magnetic fields. The fluid dynamical equations are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{mn}\frac{\partial p}{\partial x} = -\frac{e}{m}E, \tag{4.2}$$

$$\frac{\partial p}{\partial t} + 3\frac{\partial}{\partial x}(pu) - 2u\frac{\partial p}{\partial x} = 0, \tag{4.3}$$

$$\partial E/\partial t = 4\pi enu, \tag{4.4}$$

where n is the electron number density, m is the mass of an electron, u is the velocity, p is the pressure, and E is the electric field.

To determine an approximate solution for small but finite amplitude waves we assume that

$$n = n_0 + \sum_{s=1}^{3} \epsilon^s n_s(\xi, \tau, \theta) + O(\epsilon^4), \tag{4.5}$$

$$u = \sum_{s=1}^{3} \epsilon^s u_s(\xi, \tau, \theta) + O(\epsilon^4), \tag{4.6}$$

$$p = p_0 + \sum_{s=1}^{3} \epsilon^s p_s(\xi, \tau, \theta) + O(\epsilon^4), \tag{4.7}$$

$$E = \sum_{s=1}^{3} \epsilon^s E_s(\xi, \tau, \theta) + O(\epsilon^4), \tag{4.8}$$

where ξ, τ, and θ as well as the derivatives are given by (2.12) and (2.13). Here ϵ is a small but finite quantity of the order of the amplitude of oscillation; it is introduced for convenience to keep track of the ordering and it will be set equal to unity in the final solution.

Substituting (4.5)–(4.8), (2.12) and (2.13) into (4.1)–(4.4), and equating coefficients of like powers of ϵ, we get equations to determine the different perturbation quantities. The solution of the first-order problem is taken to be

$$E_1 = A(\xi, \tau) e^{i\theta} + \bar{A}(\xi, \tau) e^{-i\theta}, \tag{4.9}$$

$$u_1 = \frac{i\omega}{4\pi e n_0} [A(\xi, \tau) e^{i\theta} - \bar{A}(\xi, \tau) e^{-i\theta}], \tag{4.10}$$

$$n_1 = -\frac{ik}{4\pi e} [A(\xi, \tau) e^{i\theta} - \bar{A}(\xi, \tau) e^{-i\theta}], \tag{4.11}$$

$$p_1 = \frac{3ikp_0}{4\pi e n_0} [A(\xi, \tau) e^{i\theta} - \bar{A}(\xi, \tau) e^{-i\theta}], \tag{4.12}$$

where ω and k satisfy the dispersion relationship

$$\omega^2 = \omega_p^2 + k^2 V_e^2. \tag{4.13}$$

Here, ω_p and V_e are the plasma frequency and the thermal speed and given by

$$\omega_p^2 = 4\pi n_0 e^2/m, \quad V_e^2 = 3p_0/mn_0. \tag{4.14}$$

Then, the solution of the second-order problem is

$$E_2 = \frac{k\omega(3\omega_p^2 + 4k^2 V_e^2)}{12 i\pi e n_0 \omega_p^2} (A^2 e^{2i\theta} - \bar{A}^2 e^{-2i\theta}), \tag{4.15}$$

$$u_2 = -\frac{k\omega}{8\pi^2 e^2 n_0} A\bar{A} - \frac{k\omega(3\omega_p^2 + 8k^2 V_e^2)}{48\pi^2 e^2 n_0^2 \omega_p^2} (A^2 e^{2i\theta} + \bar{A}^2 e^{-2i\theta}), \tag{4.16}$$

$$p_2 = -\frac{k^2 p_0(9\omega_p^2 + 8k^2 V_e^2)}{16\pi^2 e^2 n_0^2 \omega_p^2} (A^2 e^{2i\theta} + \bar{A}^2 e^{-2i\theta}), \tag{4.17}$$

$$n_2 = -\frac{k^2(3\omega_p^2 + 4k^2 V_e^2)}{24\pi^2 e^2 n_0^2 \omega_p^2} (A^2 e^{2i\theta} + \bar{A}^2 e^{-2i\theta}). \tag{4.18}$$

Using the first- and second-order solutions, we can combine the third-order equations to give

$$\omega\omega_p^2\left(\frac{\partial^2 E_3}{\partial\theta^2}+E_3\right) = i\{\omega_p^2 A_\tau+\omega(A\omega)_\tau+V_e^2k[(Ak)_\tau+(A\omega)_\xi] + \omega V_e^2(Ak)_\xi - 8i\omega^2 JA^2\bar{A}\}e^{i\theta}+\mathrm{CC}+\mathrm{NSPT}, \quad (4.19)$$

where
$$J = -\frac{k^4V_e^2(3\omega_p^2+8k^2V_e^2)}{96\pi^2e^2n_0^2\omega_p^2\omega}. \quad (4.20)$$

Annihilating the secular producing terms on the right-hand side of (4.19), and using the relationships $k_\tau = -\omega_\xi = -c_g k_\xi$ and $c_g = d\omega/dk$, we get

$$2\frac{\partial A}{\partial\tau}+2c_g\frac{\partial A}{\partial\xi}+c_g' A\frac{\partial k}{\partial\xi} = 8iJA^2\bar{A}, \quad (4.21)$$

where $c_g' = dc_g/dk$. Letting $A = \frac{1}{2}a\exp i\beta$ with real a and β in (4.21), and separating the real and imaginary parts, we obtain

$$\frac{\partial a^2}{\partial\tau}+\frac{\partial}{\partial\xi}(c_g a^2) = 0, \quad (4.22)$$

$$\frac{\partial\beta}{\partial\tau}+c_g\frac{\partial\beta}{\partial\xi} = Ja^2. \quad (4.23)$$

If ω and k are taken to be constants, then (4.21) reduces to the equations of Tidman & Stainer (1965) and Nayfeh (1965). Equations (4.22) and (4.23) have the same form as (2.37) and (2.38) obtained for the interaction of capillary and gravity waves, and (3.17) and (3.18) obtained for waves on the surface of a circular liquid jet. These equation are discussed in the next section.

5. Discussion

In our treatment of non-linear dispersive waves (i) on the interface between a subsonic gas and a liquid, (ii) on the surface of a circular column of liquid, and (iii) in a hot electron plasma, we found that the wave-numbers, amplitudes, and phases are governed by

$$\frac{\partial k}{\partial t}+c_g\frac{\partial k}{\partial x} = 0, \quad (5.1)$$

$$\frac{\partial a^2}{\partial t}+\frac{\partial}{\partial x}(c_g a^2) = 0, \quad (5.2)$$

$$\frac{\partial\beta}{\partial t}+c_g\frac{\partial\beta}{\partial x} = J\epsilon^2a^2, \quad (5.3)$$

where c_g is the group velocity and J is a known function of k and the parameters of the problem considered. Of course, c_g and J are not the same for the three problems considered.

Although the three physical problems considered in this paper represent weakly non-linear waves, they seem to be rather unrelated problems. However, when we treated them by the same mathematical technique, we found out that the equations governing the wave-number, amplitude and the phase are formally given by the same equations (5.1)–(5.3). This result lead us to conclude that these problems represent the same physical phenomenon. To emphasize this fact, we felt that we should publish all of these results in one paper rather than in three separate papers.

To explain physically why these three problems should be represented formally by the same equations, we notice that (5.1) represents the conservation of the waves. Equation (5.2) states that the energy represented by the square of the amplitude propagates with the group velocity c_g along the characteristics

$$dx/dt = c_g(k). \tag{5.4}$$

This is so because, if Δx is a distance between two of these characteristics,

$$\frac{d}{dt}(a^2\Delta x) = \Delta x\frac{da^2}{dt}+a^2\Delta\frac{dx}{dt} = \Delta x\left[\frac{da^2}{dt}+a^2c_g'\frac{\partial k}{\partial x}\right] = 0, \tag{5.5}$$

because (5.2) can be written as

$$da^2/dt = -c_g' k_x a^2 \tag{5.6}$$

along the characteristics (5.4). To interpret (5.3), we observe that in the non-linear case, the dispersion relation can be written as

$$\omega = f(k; a^2). \tag{5.7}$$

For small a one can expand this equation to obtain

$$\omega = f_0(k)+f_2(k)a^2+\ldots, \tag{5.8}$$

where $\omega_0 = f_0(k_0)$ is the linear dispersion relationship. If we assume that

$$\omega = \omega_0+\Delta\omega, \quad k = k_0+\Delta k \tag{5.9}$$

in (5.8), and keep linear terms only, we have

$$\Delta\omega = c_g\Delta k+f_2(k_0)a^2, \tag{5.10}$$

where $c_g = f_0'(k_0)$ is the group velocity. If we introduce a phase function $\beta(x,t)$ such that

$$\Delta\omega = -\partial\beta/\partial t, \quad \Delta k = \partial\beta/\partial x, \tag{5.11}$$

then (5.10) will have the form (5.3). Therefore (5.3) represents the second-order term in the dispersion relationship.

Equations (5.1) and (5.2) show that to second order k and a are not altered by the non-linear motion. However, the effect of the non-linear motion modified β as seen from (5.3). These results indicate that, to determine (5.1)–(5.3) for any weakly non-linear dispersive non-dissipative wave problem, we do not need to carry out an analysis similar to those presented in §§2–4. From a linear analysis, we get the dispersion relationship $\omega_0 = f_0(k_0)$ and the group velocity $c_g = d\omega_0/dk_0$ which determines (5.1) and (5.2). To write down (5.3), we need only to determine J. This can be done by carrying out an analysis for constant ω, k, and a to obtain either $d\beta/dt$ or $d\beta/dx$ because

$$J = (1/\epsilon^2a^2)\,[d\beta/dt,\ c_g d\beta/dx]$$

from (5.3).

To solve (5.1)–(5.3), we solve (5.1) first to obtain k = a constant and hence ω = a constant along the characteristics $dx/dt = c_g(k)$, which are straight lines in the (x, t) plane as a consequence. By writing (5.2) and (5.3) as in (5.6) and

$$d\beta/dt = -J\epsilon^2 a^2 \tag{5.12}$$

along the same characteristics, we can compute a^2 and then β.

For a constant ω and hence a constant k, (5.3) reduces to

$$\frac{\partial a^2}{\partial t} + c_g \frac{\partial a^2}{\partial x} = 0. \tag{5.13}$$

The solution of (5.13) is

$$a^2 = F(x - c_g t), \tag{5.14}$$

where F is determined from the initial conditions. Since $J = J(k)$ is a constant, the solution of (5.3) is

$$\beta = \tfrac{1}{2}\epsilon^2 (J/c_g)\,(x + c_g t)\,F(x - c_g t) + \epsilon^2 G(x - c_g t), \tag{5.15}$$

where G is determined also from the initial conditions. A similar solution to (5.14) and (5.15) was obtained by Nayfeh (1971*a*) for the case of waves on the surface of a liquid of infinite depth.

If the initial conditions are such that a and β are independent of position, in addition to ω and k being constants, then a = a constant, and

$$\beta = J\epsilon^2 a^2 t. \tag{5.16}$$

This yields a frequency shift. If a and β are time-independent,

$$\beta = J\epsilon^2 a^2 x \tag{5.17}$$

and the non-linear motion produces a wave-number shift.

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Appendix

$$p_1 = -M^2\left[\frac{1}{2}\left(\frac{\gamma+1}{m^2} - \gamma + 3\right) f_3 + \tfrac{1}{8}(\gamma+1)(2\gamma-3)\frac{M^4}{m^3} + \tfrac{1}{8}(\gamma-1)\frac{M^2}{m}\left(\frac{1}{m^2}+3\right) - \frac{3m^4+2m^2-1}{4m^3}\right]; \tag{A 1}$$

$$p_2 = -\frac{1}{8}\frac{M^6}{m^2}(\gamma+1)\left(\frac{\gamma+1}{m^2} - \gamma + 3\right); \tag{A 2}$$

$$p_3 = \frac{1}{8}\frac{\omega}{k}(4f_1C_1 + 8f_2C_2 + 3); \tag{A 3}$$

$$p_4 = \left(2m - \frac{1}{m}\right) f_3 + \frac{1}{2}\left(\frac{2}{m} - m\right) f_1 - \tfrac{1}{4} - \tfrac{1}{4}(\gamma+1)\frac{M^4}{m^2} - \tfrac{1}{8}m^2; \tag{A 4}$$

$$p_5 = [(1 - C_1C_2)f_2 + \tfrac{1}{2}f_1 - \tfrac{5}{8}C_1]\frac{\omega^2}{k^2} + \tfrac{3}{8}k^2; \tag{A 5}$$

$$p_6 = \tfrac{1}{16}m\left[(\gamma+1)\frac{M^4}{m^3} - 10m - 2\frac{M^2}{m^2}\left(\frac{3}{m^2}+1\right) - 8f_1\right]. \tag{A 6}$$

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